Technical Comments

Comment on "Reduction of Structural Frequency Equations"

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A METHOD for improving the accuracy of some of the calculations associated with approximations which reduce the number of degrees of freedom in matrix formulations of the eigenvalue problems of small vibrations has been presented by R. L. Kidder.[1] Such approximations have been put forward by a number of authors.[2-5] The techniques are called by various names such as "reduction," "condensation," and "economization." Irons[3] and Fried[4] have pointed out that such techniques generally constitute applications of the Rayleigh and Rayleigh-Ritz methods to the matrix eigenvalue formulation for structural vibration problems. Guyan[2] and Zienkiewicz[5] derived the reduction method on the basis of contragradient transformations of the coordinates in the equations of motion. These transformations have the effect of maintaining the invariance of the quadratic forms for potential and kinetic energies. Since the Rayleigh-Ritz method generally also preserves the invariance of the quadratic forms for potential and kinetic energies, the results obtained from the two methods of derivation are equivalent.

Kidder's analysis parallels Guyan's treatment of the problem in that he partitions the mass and stiffness matrices corresponding to primary and secondary (or "master" and "slave") displacement vectors, $\{x_1\}$ and $\{x_2\}$, respectively, to obtain the equations of motion

$$\left(-\omega^2\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix}+\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{bmatrix}\right)\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=0 \qquad (1)$$

where the M_{ij} ank K_{ij} are submatrices.

However, rather than relating his approach to a Rayleigh-Ritz or contragradient transformation formulation, Kidder uses a matrix series expansion of $\{x_2\}$ in terms of $\{x_1\}$ to derive both the method for finding the eigenvalues and eigenvectors $\{x_1\}$ of the reduced problem and the back transformation giving $\{x_2\}$ in terms of $\{x_1\}$. Kidder's matrix series expansion is

$$(-\omega^2 M_{22}+K_{22})^{-1} = K_{22}^{-1}+\omega^2 K_{22}^{-1}M_{22}K_{22}^{-1}+\cdots \qquad (2)$$

and he retains only terms of order ω^2 and lower. Unfortunately, this series does not converge[6] except for values of ω^2 which are lower than any eigenvalue $\omega_m{}^2$ of the eigenvalue problem

$$(-\omega^2 M_{22}+K_{22})\{x_2\} = 0 \qquad (3)$$

Thus, unless the frequency ω^2 is less than *all* of the natural frequencies of the dynamical system comprising the degrees of freedom associated with the secondary displacement vector, Kidder's derivation of Guyan's reduction method and his formula for the back transformation are both invalid. However, Kidder's derivation does not change Guyan's reduction method which can be formulated correctly by the Rayleigh-Ritz or contragradient transformation approaches. On the other hand, Kidder's correction of the back-transformation may lead to substantial errors except when the conditions for convergence of the matrix series expansion cited are met.

The Rayleigh-Ritz formulation of the small-amplitude vibration problem may be employed to examine the question of higher order corrections to the coordinates x_2 (which, in Guyan's approach, are specified in the first approximation by static coupling only). The equations of motion of a dynamical system may be the result of direct application of Newton's laws or the less direct Lagrangian or Hamiltonian statements of these laws and they may embody assumed continuous or discrete generalized coordinates, the latter perhaps following from finite difference or finite element approaches.

Regardless of how arrived at, the problem reduces to finding the eigenvalues and eigenfunction of a finite set of algebraic linear homogeneous equations corresponding to Eq. (1). For compactness of notation, the matrix form is convenient, but nothing we shall use requires any of the finer points of the theory of matrices. Guyan's reduction scheme depends on making the assumption that the coordinates, x_2, are statically coupled to the coordinates, x_1, i.e.,

$$\{x_2\} = K_{22}^{-1}K_{21}\{x_1\} \qquad (4)$$

This assumed relationship reduces the matrix eigenvalue problem to one involving the coordinates $\{x_1\}$ only.

If, however, we wish to retain the coordinates $\{x_2\}$ in a form which permits calculation of higher order corrections arising from dynamic coupling, it is convenient to write

$$\{x_2\} = -K_{22}^{-1}K_{21}\{x_1\}+\{x_2'\} \qquad (5)$$

Then Eq. (2) becomes

$$[-\omega^2(M_{11}-M_{12}K_{22}^{-1}K_{21})+(K_{11}-K_{12}K_{22}^{-1}K_{21})]\{x_1\}+ [-\omega^2 M_{12}+K_{12}]\{x_2'\}=0 \qquad (6)$$
$$[-\omega^2(M_{21}-M_{22}K_{22}^{-1}K_{21})]\{x_1\}+[-\omega^2 M'_{22}K_{22}]\{x_2'\}=0$$

It can be seen that the substitution of Eq. (5) has had the effect of eliminating any static coupling of $\{x_1\}$ from the stiffness matrix in the second equation in $\{x_1\}$ and $\{x_2'\}$.

To put these last two equations in symmetrical Rayleigh-Ritz form, the second equation is multiplied by $-K_{12}K_{22}^{-1}$ and added to the first. This results in

$$[-\omega^2 M_{11}{}^R+K_{11}{}^R]\{x_1\}+ [-\omega^2(M_{12}-K_{12}K_{22}^{-1}M_{22})]\{x_2'\}=0$$
$$[-\omega^2(M_{21}-M_{22}K_{22}^{-1}K_{21})]\{x_1\}+ [-\omega^2 M_{22}+K_{22}]\{x_2'\}=0 \qquad (7)$$

where

$$M_{11}{}^R = M_{11}+K_{12}K_{22}^{-1}M_{22}K_{22}^{-1}K_{21}- K_{12}K_{22}^{-1}M_{21}-M_{12}K_{22}^{-1}K_{12} \qquad (8)$$

and

$$K_{11}{}^R = K_{11}-K_{12}K_{22}^{-1}K_{21}$$

Static coupling is now completely eliminated between the $\{x_1\}$ and $\{x_2\}$ coordinates. The terms $M_{11}{}^R$ and $K_{11}{}^R$ may be seen to correspond to the formulas for reduction of matrices given by Guyan,[2] Kidder[1] and others. The reduction of Eq. (1) to

$$(-\omega^2 M_{11}{}^R+K_{11}{}^R)\{x_1\}=0 \qquad (9)$$

involves the assumption that the dynamic coupling between $\{x_2'\}$ and $\{x_1\}$ may be neglected. Using the rudimentary methods of small perturbation theory,[7] it can be seen that if the coupling terms are small, then at an eigenvalue (natural frequency) of Eq. (9) $\Omega_n{}^2$ with corresponding eigenvector $\{x_{1n}\}$ a first-order correction $\{x'_{2n}\}$ is given from the second of Eqs. (7) by

$$\{x'_{2n}\} = -\Omega_n{}^2[-\Omega_n{}^2 M_{22}+K_{22}]^{-1}\times [M_{21}-M_{22}K_{22}^{-1}K_{21}]\{x_{1n}\} \qquad (10)$$

and the correction to $\Omega_n{}^2$, which is of second order in the coupling term, is given by

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$$\Delta\Omega_n^2 = -\Omega_n^4/M_{1n}\{x_{1n}\}^T[M_{12}-K_{12}K_{22}^{-1}M_{22}] \times [-\Omega_n^2 M_{22}+K_{22}]^{-1}[M_{21}-M_{22}K_{22}^{-1}K_{21}]\{x_{1n}\} \quad (11)$$

where

$$M_{1n} = \{x_{1n}\}^T[M_{11}]\{x_{1n}\}$$

In these equations for $\{x'_{2n}\}$ and $\Delta\Omega_n^2$, it is not permissible to use a power series expansion in ω^2 for the inversion of the matrix unless Ω_n^2 is less than any eigenvalue of the system

$$(-\omega^2 M_{22}+K_{22})\{x_2'\} = 0 \quad (12)$$

Except for the special cases when M_{22} and K_{22} are diagonal or when $(M_{12}-K_{12}K_{22}^{-1}M_{22}) = 0$ (in which case the coupling coefficients vanish), the solution for $\{x'_{2n}\}$ requires inversion of the matrix, $(-\Omega_n^2 M_{22}+K_{22})^{-1}$.

If all of the eigenvalues and eigenvectors of Eq. (12) are known, then the inverse matrix may be written in the well-known eigenfunction expansion form and Eq. (10) becomes

$$\{x'_{2n}\} = \sum_m \frac{\{Y_m\}\{Y_m\}^T\{F_n\}}{M_{2m}\omega_m^2(1-\Omega_n^2/\omega_m^2)} \quad (13)$$

where ω_m^2 and $\{Y_m\}$ are, respectively, the mth eigenvalue (assumed to be distinct) and eigenvector of Eq. (12). M_{2m} is given by $M_{2m} = \{Y_m\}^T[M_{22}]\{Y_m\}$ and F_n is the column matrix

$$\{F_n\} = [M_{21}-M_{22}K_{22}^{-1}K_{21}]\{x_{1n}\} \quad (14)$$

Term by term, it can be seen from Eq. (13) that the power series representation of the "resonance" denominator is

$$1/1-\Omega_n^2/\omega_m^2 = 1+\Omega_n^2/\omega_m^2+\Omega_n^4/\omega_m^4+\cdots \quad (15)$$

and this is convergent only if $\Omega_n^2/\omega_m^2 < 1.0$. Thus, the power series representation can be used only if Ω_n^2/ω_m^2 is less than one. In other cases it may led to "corrections" not only of the wrong magnitude but of the wrong sign. For the lowest eigenvalues, Ω_n^2, it may happen that none of the ω_m are exceeded so that the power series approximation may be valid. If only one or a few of the ω_n^2's are exceeded, the corresponding eigenvalues and eigenvectors may be found by iteration and used to form the proper terms in the eigenfunction expansion, Eq. (13), while the terms corresponding to higher eigenvalues of ω_n^2 are approximated by a power series representation similar to that proposed by Kidder. The details of this procedure have been presented in Ref. 8 for a similar eigenvalue problem in aeroelasticity.

Of course the approximation process of perturbation theory described above may be continued formally to higher order terms although this may be a long process if the perturbation series is poorly convergent. Also, instead of calculating the second-order correction to frequency from Eq. (11), the eigenvectors $\{x_{1n}\}$ and $\{x_{2n}\}+\{x'_{2n}\}$ may be used to substitute for $\{x_1\}$ and $\{x_2\}$, respectively, in the Rayleigh quotient corresponding to Eq. (1) to obtain new approximations for the frequencies. The latter procedure will usually be more accurate, at least for the lower frequencies.

References

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Reply by Author to A. H. Flax

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The author wishes to thank A. H. Flax for his comments and for calling attention to the convergence requirement on the series expansion

$$(-\omega^2 M_{22}+K_{22})^{-1} = K_{22}^{-1}+\omega^2 K_{22}^{-1}M_{22}K_{22}^{-1}+\cdots \quad (1)$$

wherein Eq. (1) will always converge for frequencies ω which are less than the smallest frequency $\bar{\omega}$ obtained from the eigenproblem

$$(-\bar{\omega}^2 M_{22}+K_{22})\{x_2\} = \{0\} \quad (2)$$

However, the author must take exception to Flax's erroneous assertion that the reduction method of Eqs. (5–7) of Ref. 1 was not derived correctly. The point Flax has overlooked is that the reduction method of an eigenproblem is an approximation technique to obtain the *lowest* modes and frequencies of the total structure and avoid the computational difficulties of solving a large eigenproblem. For the reduction method to be properly applied, it is implicitly assumed that the degrees of freedom retained must be those associated with the lowest modes and frequencies of the structure and those eliminated with the higher modes and frequencies. If this is not accomplished, the reduction procedure yields worthless results.

For a large complex structure where the degrees of freedom corresponding to local modes of low frequency have been inadvertently eliminated, it is doubtful that the results of the reduction procedure could be adequately refined, using the perturbation technique presented by Flax, to recover these local modes and their corresponding frequencies. Therefore, when the reduction method is used properly, the solution frequencies of interest will automatically satisfy the convergence requirement a priori; the expansion of Eq. (1) and the simplified back-transformation expression

$$\{x_2\} = -(K_{22}^{-1}+\omega^2 K_{22}^{-1}M_{22}K_{22}^{-1})(-\omega^2 M_{21}+K_{21})\{x_1\} \quad (3)$$

will both be valid approximations.

Usually, the analyst can only rely on his engineering judgment and expertise to make the proper selection as to which degrees of freedom to retain and which to eliminate. But now, by investigating the approximations involved in the series expansion of Eq. (1), there is a definite criterion to be used in making a selection. That is, in order to obtain the lowest frequencies in the reduced problem, eliminate those degrees of freedom which maximize the lowest frequency of the convergence requirement Eq. (2). For a complicated structure it may be difficult to determine how to implement this selection procedure, but, at least, there now is a guideline to follow. It should be pointed out, as shown by Eq. (2), that both the mass (kinetic energy) and the stiffness (potential energy) coefficients must be considered when selecting degrees of freedom to eliminate. To illustrate the reduction method and the back-transformation scheme of Eq. (3) the following example problem is presented.

Figure 1 depicts a prismatic beam simply supported; it is desired to investigate the small lateral vibrations of this beam in the x-y plane. The beam is characterized by the parameters A, E, I, ρ, and L which are the cross sectional area, modulus of elasticity, moment of inertia, mass density, and half-length of the beam, respectively. The beam has been discretized to three node points with four degrees of freedom, a node at each end, each with a rotational degree of freedom, and a node at midspan with a lateral translational and a rotational degree of freedom. The problem is to be reduced to two degrees of freedom by

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